

Generalized entanglement constraints in multi-qubit systems in terms of Tsallis entropy

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We provide generalized entanglement constraints in multi-qubit systems in terms of Tsallis entropy. Using quantum Tsallis entropy of order q , we first provide a generalized monogamy inequality of multi-qubit entanglement for $q = 2$ or 3 . This generalization encapsulates multi-qubit CKW-type inequality as a special case. We further provide a generalized polygamy inequality of multi-qubit entanglement in terms of Tsallis- q entropy for $1 \leq q \leq 2$ or $3 \leq q \leq 4$, which also contains the multi-qubit polygamy inequality as a special case.

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I. INTRODUCTION

Quantum Tsallis entropy is a one-parameter generalization of von Neumann entropy with respect to a non-negative real parameter q [1, 2]. Tsallis entropy is used in many areas of quantum information theory; Tsallis entropy can be used to characterize classical statistical correlations inherent in quantum states [3], and it provides some conditions for separability of quantum states [4–6]. There are also discussions about using the non-extensive statistical mechanics to describe quantum entanglement in terms of Tsallis entropy [7].

As a function defined on the set of density matrices, Tsallis entropy is concave for all $q > 0$, which plays an important role in quantum entanglement theory. Because the concavity of Tsallis entropy assures the property of *entanglement monotone* [8], it can be used to construct a faithful entanglement measure, which does not increase under *local quantum operations and classical communication* (LOCC).

One distinct property of quantum entanglement from other classical correlations is that multi-party entanglement cannot be freely shared among the parties. This restricted shareability of entanglement in multi-party quantum systems is known as *monogamy of entanglement* (MoE) [9, 10]. MoE is a key ingredient for secure quantum cryptography [11, 12], and it also plays an important role in condensed-matter physics such as the N -representability problem for fermions [13].

Using *concurrence* [14] as a bipartite entanglement measure, Coffman-Kundu-Wootters (CKW) provided a mathematical characterization of MoE in three-qubit systems as an inequality [15], which was generalized for arbitrary multi-qubit systems [16]. As a dual concept of MoE, a *polygamy* inequality of multi-qubit entanglement was established in terms of *Concurrence of Assistance* (CoA). Later, it was shown that the monogamy and polygamy in-

equalities of multi-qubit entanglement can also be established by using other entropy-based entanglement measures such as Rényi, Tsallis and unified entropies [17–19].

Recently, a different kind of monogamous relation in multi-qubit entanglement was proposed by using concurrence and CoA [20]. Whereas the CKW-type monogamy inequalities of multi-qubit entanglement provide a lower bound of bipartite entanglement between one qubit subsystem and the rest qubits in terms of two-qubit entanglement, the new kind of monogamy relations in [20] provide bounds of bipartite entanglement between a two-qubit subsystem and the rest in multi-qubit systems in terms of two-qubit concurrence and CoA.

Here, we provide generalized entanglement constraints in multi-qubit systems in terms of Tsallis entropy for a selective choice of the real parameter q . Using quantum Tsallis entropy of order q , namely *Tsallis- q entropy*, we first show that the CKW-type monogamy inequality of multi-qubit entanglement can have a generalized form for $q = 2$ or 3 . This generalized monogamy inequality encapsulates multi-qubit CKW-type monogamy inequality as a special case. We further provide a generalized polygamy inequality of multi-qubit entanglement in terms of Tsallis- q entropy for $1 \leq q \leq 2$ or $3 \leq q \leq 4$, which also contains multi-qubit polygamy inequality as a special case.

This paper is organized as follows. In Sec. II A, we recall the definition of Tsallis- q entropy, and the bipartite entanglement measure based on Tsallis entropy, namely Tsallis- q entanglement as well as its dual quantity, Tsallis- q entanglement of assistance (TEoA). In Sec. II B, we review the analytic evaluations of Tsallis- q entanglement and TEoA in two-qubit systems based on their functional relations with concurrence, and we further review the monogamy and polygamy inequalities of multi-qubit entanglement in terms of Tsallis- q entanglement and TEoA in Sec. III. In Sec. IV, we provide generalized monogamy and polygamy inequalities of multi-qubit entanglement in terms of Tsallis- q entanglement and TEoA, and we summarize our results in Sec. V.

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II. TSALLIS- q ENTANGLEMENT

A. Definition

Using a generalized logarithmic function with respect to the parameter q ,

$$\ln_q x = \frac{x^{1-q} - 1}{1 - q}, \quad (1)$$

quantum Tsallis- q entropy for a quantum state ρ is defined as

$$S_q(\rho) = -\text{tr} \rho^q \ln_q \rho = \frac{1 - \text{tr}(\rho^q)}{q - 1} \quad (2)$$

for $q > 0$, $q \neq 1$ [2]. Although the quantum Tsallis- q entropy has a singularity at $q = 1$, it converges to von Neumann entropy when q tends to 1 [21],

$$\lim_{q \rightarrow 1} S_q(\rho) = -\text{tr} \rho \ln \rho = S(\rho). \quad (3)$$

Based on Tsallis- q entropy, a class of bipartite entanglement measures was introduced; for a bipartite pure state $|\psi\rangle_{AB}$ and each $q > 0$, its *Tsallis- q entanglement* [18] is

$$\mathcal{T}_q(|\psi\rangle_{A|B}) = S_q(\rho_A), \quad (4)$$

where $\rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi|$ is the reduced density matrix of $|\psi\rangle_{AB}$ onto subsystem A . For a bipartite mixed state ρ_{AB} , its Tsallis- q entanglement is defined via convex-roof extension,

$$\mathcal{T}_q(\rho_{A|B}) = \min \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{A|B}), \quad (5)$$

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$.

Because Tsallis- q entropy converges to von Neumann entropy when q tends to 1, we have

$$\lim_{q \rightarrow 1} \mathcal{T}_q(\rho_{A|B}) = E_f(\rho_{A|B}), \quad (6)$$

where $E_f(\rho_{AB})$ is the EoF [22] of ρ_{AB} defined as

$$E_f(\rho_{A|B}) = \min \sum_i p_i S(\rho_A^i), \quad (7)$$

with the minimization over all possible pure state decompositions of ρ_{AB} ,

$$\rho_{AB} = \sum_i p_i |\psi^i\rangle_{AB} \langle \psi^i|, \quad (8)$$

and $\text{tr}_B |\psi^i\rangle_{AB} \langle \psi^i| = \rho_A^i$. In other words, Tsallis- q entanglement is one-parameter generalization of EoF, and the singularity of $\mathcal{T}_q(\rho_{AB})$ at $q = 1$ can be replaced by $E_f(\rho_{AB})$.

As a dual quantity to Tsallis- q entanglement, *Tsallis- q entanglement of Assistance* (TEoA) was defined as [18]

$$\mathcal{T}_q^a(\rho_{A|B}) := \max \sum_i p_i \mathcal{T}_q(|\psi_i\rangle_{A|B}), \quad (9)$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB} . Similarly, we have

$$\lim_{q \rightarrow 1} \mathcal{T}_q^a(\rho_{A|B}) = E^a(\rho_{A|B}), \quad (10)$$

where $E^a(\rho_{A|B})$ is the *Entanglement of Assistance* (EoA) of ρ_{AB} defined as [23]

$$E^a(\rho_{A|B}) = \max \sum_i p_i S(\rho_A^i). \quad (11)$$

with the maximization over all possible pure state decompositions of ρ_{AB} .

B. Functional relation with concurrence in two-qubit systems

For any bipartite pure state $|\psi\rangle_{AB}$, its concurrence is defined as [14]

$$\mathcal{C}(|\psi\rangle_{A|B}) = \sqrt{2(1 - \text{tr} \rho_A^2)}, \quad (12)$$

where $\rho_A = \text{tr}_B (|\psi\rangle_{AB} \langle \psi|)$. For a mixed state ρ_{AB} , its concurrence and concurrence of assistance (CoA) are defined as

$$\mathcal{C}(\rho_{A|B}) = \min \sum_k p_k \mathcal{C}(|\psi_k\rangle_{A|B}), \quad (13)$$

and

$$\mathcal{C}^a(\rho_{A|B}) = \max \sum_k p_k \mathcal{C}(|\psi_k\rangle_{A|B}), \quad (14)$$

respectively, where the minimum and maximum are taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k |\psi_k\rangle_{AB} \langle \psi_k|$.

For two-qubit systems, concurrence and CoA are known to have analytic formulae [14]; for any two-qubit state ρ_{AB} ,

$$\mathcal{C}(\rho_{A|B}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (15)$$

$$\mathcal{C}^a(\rho_{A|B}) = \sum_{i=1}^4 \lambda_i, \quad (16)$$

where λ_i 's are the eigenvalues, in decreasing order, of $\sqrt{\sqrt{\rho_{AB}} \tilde{\rho}_{AB} \sqrt{\rho_{AB}}}$ and $\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y \rho_{AB}^* \sigma_y \otimes \sigma_y$ with the Pauli operator σ_y .

Later, it was shown that there is a functional relation between concurrence and Tsallis- q entanglement in

two-qubit systems [18]. For any two-qubit state ρ_{AB} (or bipartite pure state with Schmidt-rank 2), we have

$$\mathcal{T}_q(\rho_{A|B}) = f_q(\mathcal{C}(\rho_{A|B})), \quad (17)$$

for $1 \leq q \leq 4$ where $f_q(x)$ is a monotonically increasing convex function defined as

$$f_q(x) = \frac{1}{q-1} \left[1 - \left(\frac{1 + \sqrt{1-x^2}}{2} \right)^q - \left(\frac{1 - \sqrt{1-x^2}}{2} \right)^q \right] \quad (18)$$

on $0 \leq x \leq 1$ [24].

Here we note that the analytic evaluation of concurrence in Eq. (15) together with the functional relations in Eq. (17) provides us with an analytic formula of Tsallis entanglement in two-qubit systems. Moreover, the monotonicity and convexity of $f_q(x)$ for $1 \leq q \leq 4$ also provide an analytic lower bound of TEOA,

$$\mathcal{T}_q^a(\rho_{A|B}) \geq f_q(\mathcal{C}^a(\rho_{A|B})), \quad (19)$$

where the equality holds $q = 2$ or 3 [18].

III. MULTI-QUBIT ENTANGLEMENT CONSTRAINTS IN TERMS OF TSALLIS ENTROPY

The monogamy of a multi-qubit entanglement was shown to have a mathematical characterization as an inequality; for a multi-qubit state $\rho_{A_1 A_2 \dots A_n}$,

$$\mathcal{C}(\rho_{A_1|A_2 \dots A_n})^2 \geq \mathcal{C}(\rho_{A_1|A_2})^2 + \dots + \mathcal{C}(\rho_{A_1|A_n})^2, \quad (20)$$

where $\mathcal{C}(\rho_{A_1|A_2 \dots A_n})$ is the concurrence of $\rho_{A_1 A_2 \dots A_n}$ with respect to the bipartition between A_1 and the other qubits, and $\mathcal{C}(\rho_{A_1|A_i})$ is the concurrence of the two-qubit reduced density matrix $\rho_{A_1 A_i}$ for $i = 2, \dots, n$ [15, 16]. Moreover, the *polygamy* (or dual monogamy) inequality of multi-qubit entanglement was also established using CoA [25] as

$$(\mathcal{C}^a(\rho_{A_1|A_2 \dots A_n}))^2 \leq (\mathcal{C}^a(\rho_{A_1|A_2}))^2 + \dots + (\mathcal{C}^a(\rho_{A_1|A_n}))^2, \quad (21)$$

where $\mathcal{C}^a(\rho_{A_1|A_2 \dots A_n})$ is the CoA of $\rho_{A_1 A_2 \dots A_n}$ with respect to the bipartition between A_1 and the other qubits, and $\mathcal{C}^a(\rho_{A_1|A_i})$ is the CoA of the two-qubit reduced density matrix $\rho_{A_1 A_i}$ for $i = 2, \dots, n$.

Later, this mathematical characterization of monogamy and polygamy of multi-qubit entanglement was also proposed in terms of Tsallis entropy, which encapsulate the inequalities (20) and (21) as special cases [18]. Based on the following property of the function $f_q(x)$ in Eq. (18) for $2 \leq q \leq 3$,

$$f_q(\sqrt{x^2 + y^2}) \geq f_q(x) + f_q(y), \quad (22)$$

the Tsallis monogamy inequality of multi-qubit entanglement was proposed as

$$\mathcal{T}_q(\rho_{A_1|A_2 \dots A_n}) \geq \mathcal{T}_q(\rho_{A_1|A_2}) + \dots + \mathcal{T}_q(\rho_{A_1|A_n}), \quad (23)$$

for $2 \leq q \leq 3$.

For the case when $1 \leq q \leq 2$ or $3 \leq q \leq 4$, the function $f_q(x)$ in Eq. (18) also satisfies

$$f_q(\sqrt{x^2 + y^2}) \leq f_q(x) + f_q(y), \quad (24)$$

which leads to the Tsallis polygamy inequality

$$\mathcal{T}_q^a(\rho_{A_1|A_2 \dots A_n}) \leq \mathcal{T}_q^a(\rho_{A_1|A_2}) + \dots + \mathcal{T}_q^a(\rho_{A_1|A_n}) \quad (25)$$

for any multi-qubit state $\rho_{A_1 A_2 \dots A_n}$.

IV. GENERALIZED MULTI-QUBIT ENTANGLEMENT CONSTRAINTS IN TERMS OF TSALLIS ENTROPY

In this section, we provide generalized monogamy and polygamy inequalities of multi-qubit entanglement in terms of Tsallis entanglement and TEOA. We first recall some properties of Tsallis entropy.

Proposition 1. (*Subadditivity of Tsallis entropy*) For any bipartite quantum state ρ_{AB} with $\rho_A = \text{tr}_B \rho_{AB}$, $\rho_B = \text{tr}_A \rho_{AB}$, and $q \geq 1$, we have

$$S_q(\rho_{AB}) \leq S_q(\rho_A) + S_q(\rho_B). \quad (26)$$

Let us consider a three-party pure state $|\psi\rangle_{ABC}$ and its reduced density matrices $\rho_{BC} = \text{tr}_A |\psi\rangle_{ABC} \langle \psi|$, $\rho_B = \text{tr}_{AC} |\psi\rangle_{ABC} \langle \psi|$ and $\rho_C = \text{tr}_{AB} |\psi\rangle_{ABC} \langle \psi|$. For $q \geq 1$, Proposition 1 implies

$$S_q(\rho_{BC}) \leq S_q(\rho_B) + S_q(\rho_C). \quad (27)$$

Because $S_q(\rho_{BC}) = S_q(\rho_A)$ and $S_q(\rho_C) = S_q(\rho_{AB})$, Eq. (27) can be rewritten as

$$S_q(\rho_A) - S_q(\rho_B) \leq S_q(\rho_{AB}), \quad (28)$$

and similarly, we also have

$$S_q(\rho_B) - S_q(\rho_A) \leq S_q(\rho_{AB}). \quad (29)$$

Thus we have the following triangle inequality of Tsallis entropy

$$|S_q(\rho_A) - S_q(\rho_B)| \leq S_q(\rho_{AB}) \leq S_q(\rho_A) + S_q(\rho_B), \quad (30)$$

for any bipartite quantum state ρ_{AB} and $q \geq 1$.

Theorem 1. For $q = 2$ or 3 and any multi-qubit pure state $|\psi\rangle_{ABC_1C_2\cdots C_n}$, we have

$$\mathcal{T}_q(|\psi\rangle_{AB|C_1C_2\cdots C_n}) \geq \sum_{i=1}^n [\mathcal{T}_q(\rho_{A|C_i}) - \mathcal{T}_q^a(\rho_{B|C_i})], \quad (31)$$

where $\rho_{AB} = \text{tr}_{C_1\cdots C_n}(|\psi\rangle\langle\psi|)$, $\rho_{AC_i} = \text{tr}_{BC_1\cdots C_{i-1}C_{i+1}\cdots C_n}(|\psi\rangle\langle\psi|)$ and $\rho_{BC_i} = \text{tr}_{AC_1\cdots C_{i-1}C_{i+1}\cdots C_n}(|\psi\rangle\langle\psi|)$.

Proof. For simplicity, we sometimes denote $\mathbf{C} = \{C_1, C_2, \dots, C_n\}$. From the definition of Tsallis entanglement of $|\psi\rangle_{ABC_1C_2\cdots C_n}$ with respect to the bipartition between AB and \mathbf{C} , we have

$$\begin{aligned} \mathcal{T}_q(|\psi\rangle_{AB|\mathbf{C}}) &= S_q(\rho_{AB}) \\ &\geq S_q(\rho_A) - S_q(\rho_B) \\ &= \mathcal{T}_q(|\psi\rangle_{A|BC}) - \mathcal{T}_q(|\psi\rangle_{B|AC}), \end{aligned} \quad (32)$$

where the inequality is due to the Inequality (30).

We note that for any pure state $|\psi\rangle_{ABC}$ in a $2 \otimes 2 \otimes d$ quantum system with reduced density matrices $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$ and $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$, we have [26]

$$\mathcal{C}(|\psi\rangle_{A|BC})^2 = \mathcal{C}^a(\rho_{A|B})^2 + \mathcal{C}(\rho_{A|C})^2. \quad (33)$$

For $q = 2$ or 3 , Inequalities (22) and (24) imply that

$$f_q(\sqrt{x^2 + y^2}) = f_q(x) + f_q(y), \quad (34)$$

therefore

$$\begin{aligned} \mathcal{T}_q(|\psi\rangle_{A|BC}) &= f_q(\mathcal{C}(|\psi\rangle_{A|BC})) \\ &= f_q(\sqrt{\mathcal{C}^a(\rho_{A|B})^2 + \mathcal{C}(\rho_{A|C})^2}) \\ &= f_q(\mathcal{C}^a(\rho_{A|B})) + f_q(\mathcal{C}(\rho_{A|C})), \end{aligned} \quad (35)$$

where the last equality is due to Eq. (34). Moreover, we also have

$$\begin{aligned} \mathcal{T}_q(|\psi\rangle_{B|AC}) &= f_q(\mathcal{C}(|\psi\rangle_{B|AC})) \\ &\leq f_q\left(\sqrt{\mathcal{C}^a(\rho_{A|B})^2 + \sum_{i=1}^n \mathcal{C}^a(\rho_{B|C_i})^2}\right) \\ &= f_q(\mathcal{C}^a(\rho_{A|B})) + f_q\left(\sqrt{\sum_{i=1}^n \mathcal{C}^a(\rho_{B|C_i})^2}\right), \end{aligned} \quad (36)$$

where the first inequality is due to Inequality (21) and the monotonicity of $f_q(x)$ and the last equality is from Eq. (34).

Eq. (35) and Inequality (36) imply that

$$\begin{aligned} \mathcal{T}_q(|\psi\rangle_{A|BC}) - \mathcal{T}_q(|\psi\rangle_{B|AC}) &\geq f_q(\mathcal{C}(\rho_{A|\mathbf{C}})) - f_q\left(\sqrt{\sum_{i=1}^n \mathcal{C}^a(\rho_{B|C_i})^2}\right). \end{aligned} \quad (37)$$

Here we note that

$$\begin{aligned} f_q(\mathcal{C}(\rho_{A|\mathbf{C}})) &\geq f_q\left(\sqrt{\sum_{i=1}^n \mathcal{C}(\rho_{A|C_i})^2}\right) \\ &= \sum_{i=1}^n f_q(\mathcal{C}(\rho_{A|C_i})) \\ &= \sum_{i=1}^n \mathcal{T}_q(\rho_{A|C_i}), \end{aligned} \quad (38)$$

where the first inequality is due to Inequality (20) and the monotonicity of $f_q(x)$, the first equality is from the iterative use of Eq. (34), and the last equality is from the functional relation of two-qubit concurrence and Tsallis entanglement in Eq. (17). Moreover, we also have

$$\begin{aligned} f_q\left(\sqrt{\sum_{i=1}^n \mathcal{C}^a(\rho_{B|C_i})^2}\right) &= \sum_{i=1}^n f_q(\mathcal{C}^a(\rho_{B|C_i})) \\ &\leq \sum_{i=1}^n \mathcal{T}_q^a(\rho_{B|C_i}), \end{aligned} \quad (39)$$

where the first equality is from the iterative use of Eq. (34), and the last inequality is from Inequality (19).

From Inequalities (37), (38) and (39), we have

$$\begin{aligned} \mathcal{T}_q(|\psi\rangle_{A|BC}) - \mathcal{T}_q(|\psi\rangle_{B|AC}) &\geq \sum_{i=1}^n \mathcal{T}_q(\rho_{A|C_i}) - \sum_{i=1}^n \mathcal{T}_q^a(\rho_{B|C_i}), \end{aligned} \quad (40)$$

which, together with Inequality (32), completes the proof. \square

Theorem 1 provides a monogamy-type lower bound of multi-qubit entanglement between two-qubit subsystem AB and the other n -qubit subsystem $C_1C_2\cdots C_n$ in terms of two-qubit entanglements inherent there. For the case when one-qubit subsystem B is separable from other qubits, Inequality (31) reduces to the CKW-type monogamy inequality in (23), thus Theorem 1 provides a generalized monogamy relation of multi-qubit entanglement in terms of Tsallis entropy. The lower bound provided in Theorem 1 is analytically obtainable due to the analytic evaluation of two-qubit concurrence and CoA as well as their functional relation with Tsallis entanglement provided in Eq. (17) and Inequality (19).

Now, we present a generalized polygamy relation of multi-qubit entanglement in terms of TEoA. We first provide the following theorem, which shows a reciprocal relation of TEoA in three-party quantum systems.

Theorem 2. *For $q \geq 1$ any three-party quantum state ρ_{ABC} , we have*

$$\mathcal{T}_q^a(\rho_{A|BC}) \leq \mathcal{T}_q^a(\rho_{B|AC}) + \mathcal{T}_q^a(\rho_{C|AB}). \quad (41)$$

Proof. Let

$$\rho_{ABC} = \sum_j p_j |\psi_j\rangle_{ABC} \langle \psi_j| \quad (42)$$

be an optimal decomposition realizing $\mathcal{T}_q^a(\rho_{A|BC})$, that is,

$$\mathcal{T}_q^a(\rho_{A|BC}) = \sum_j p_j \mathcal{T}_q^a(|\psi_j\rangle_{A|BC}). \quad (43)$$

For each pure state $|\psi_j\rangle_{ABC}$ in the decomposition (42) with $\rho_{BC}^j = \text{tr}_A |\psi_j\rangle_{ABC} \langle \psi_j|$, $\rho_B^j = \text{tr}_{AC} |\psi_j\rangle_{ABC} \langle \psi_j|$ and $\rho_C^j = \text{tr}_{AB} |\psi_j\rangle_{ABC} \langle \psi_j|$, we have

$$\begin{aligned} \mathcal{T}_q(|\psi_j\rangle_{A|BC}) &= S_q(\rho_{BC}^j) \\ &\leq S_q(\rho_B^j) + S_q(\rho_C^j) \\ &= \mathcal{T}_q(|\psi_j\rangle_{B|AC}) + \mathcal{T}_q(|\psi_j\rangle_{C|AB}), \end{aligned} \quad (44)$$

where the inequality is due to the subadditivity of Tsallis entropy in Proposition 1.

Now we have

$$\begin{aligned} \mathcal{T}_q^a(\rho_{A|BC}) &= \sum_j p_j \mathcal{T}_q(|\psi_j\rangle_{A|BC}) \\ &\leq \sum_j p_j \mathcal{T}_q(|\psi_j\rangle_{B|AC}) + \sum_j p_j \mathcal{T}_q(|\psi_j\rangle_{C|AB}) \\ &\leq \mathcal{T}_q^a(\rho_{B|AC}) + \mathcal{T}_q^a(\rho_{C|AB}), \end{aligned} \quad (45)$$

where the first inequality is from Inequality (44), and the second inequality is due to the definition of TEoA. \square

Theorem 2 shows the reciprocal relation of TEoA in three-party quantum systems; the sum of two TEoA's with respect to two possible bipartition (B—AC and C—AB) always bounds the TEoA with respect to the remaining bipartition (A—BC). Moreover, the iterative use of Inequality (41) naturally leads us to the generalization of Theorem 2 into multi-party quantum systems.

Corollary 1. *For $q \geq 1$ and any multi-party quantum state $\rho_{A_1 A_2 \dots A_n}$,*

$$\mathcal{T}_q^a(\rho_{A_1|A_2 \dots A_n}) \leq \sum_{i=2}^n \mathcal{T}_q^a(\rho_{A_i|A_1 \dots \widehat{A_i} \dots A_n}), \quad (46)$$

where

$$\mathcal{T}_q^a(\rho_{A_i|A_1 \dots \widehat{A_i} \dots A_n}) = \mathcal{T}_q^a(\rho_{A_i|A_1 \dots A_{i-1} A_{i+1} \dots A_n}) \quad (47)$$

for each $i = 1, \dots, n$.

The following corollary presents a generalized polygamy relation of multi-qubit systems in terms of TEoA.

Corollary 2. *For $1 \leq q \leq 2$ or $3 \leq q \leq 4$ and any multi-qubit state $\rho_{ABC_1 C_2 \dots C_n}$, we have*

$$\begin{aligned} \mathcal{T}_q^a(\rho_{AB|C_1 C_2 \dots C_n}) &\leq 2\mathcal{T}_q^a(\rho_{A|B}) \\ &\quad + \sum_{i=1}^n [\mathcal{T}_q^a(\rho_{A|C_i}) + \mathcal{T}_q^a(\rho_{B|C_i})]. \end{aligned} \quad (48)$$

Proof. By considering $\rho_{ABC_1 C_2 \dots C_n}$ as a three-party quantum state $\rho_{AB\mathbf{C}}$ with $\mathbf{C} = C_1 C_2 \dots C_n$, Theorem 2 leads us to

$$\mathcal{T}_q^a(\rho_{AB|\mathbf{C}}) \leq \mathcal{T}_q^a(\rho_{A|BC}) + \mathcal{T}_q^a(\rho_{B|AC}). \quad (49)$$

Form the multi-qubit Tsallis polygamy inequality in (25), we have

$$\begin{aligned} \mathcal{T}_q^a(\rho_{A|BC}) &\leq \mathcal{T}_q^a(\rho_{A|B}) + \sum_{i=1}^n \mathcal{T}_q^a(\rho_{A|C_i}) \\ \mathcal{T}_q^a(\rho_{B|AC}) &\leq \mathcal{T}_q^a(\rho_{B|A}) + \sum_{i=1}^n \mathcal{T}_q^a(\rho_{B|C_i}). \end{aligned} \quad (50)$$

Inequality (49) together with Inequalities (50) lead us to Inequality (48). \square

Corollary 2 provides a polygamy-type upper bound of multi-qubit entanglement between two-qubit subsystem AB and the other n -qubit subsystem $C_1 C_2 \dots C_n$ in terms of two-qubit TEoA inherent there. For the case when one-qubit subsystem B is independent from other qubits (that is, $\rho_{ABC} = \rho_{AC} \otimes \rho_B$), Inequality (48) reduces to the Tsallis polygamy inequality in (25). In other words, Corollary 2 shows a generalized polygamy relation of multi-qubit entanglement in terms of TEoA.

V. CONCLUSION

We have provided generalized entanglement constraints in multi-qubit systems in terms of Tsallis- q entanglement and TEoA. We have shown that the CKW-type monogamy inequality of multi-qubit entanglement can have a generalized form in terms of Tsallis- q entanglement and TEoA for $q = 2$ or 3 . This generalized monogamy inequality encapsulates multi-qubit CKW-type inequality as a special case. We have further shown

a generalized polygamy inequality of multi-qubit entanglement in terms of TEoA for $1 \leq q \leq 2$ or $3 \leq q \leq 4$, which also contains multi-qubit polygamy inequality as a special case.

Whereas entanglement in bipartite quantum systems has been intensively studied with rich understanding, the situation becomes far more difficult for the case of multi-party quantum systems, and very few are known for its characterization and quantification. MoE is a fundamental property of multi-party quantum entanglement, which also provides various applications in quantum information theory. Thus, it is an important and even necessary task to characterize MoE to understand the whole picture of multi-party quantum entanglement.

Although MoE is a typical property of multipartite quantum entanglement, it is however about the relation of bipartite entanglements among the parties in multipartite systems. Thus, it is inevitable and crucial to have a proper way of quantifying bipartite entanglement for a

good description of the monogamy nature in multi-party quantum entanglement.

Our result presented here deals with Tsallis- q entropy, a one-parameter class of entropy functions and provide sufficient conditions on the choice of the parameter q for generalized monogamy and polygamy relations of multi-qubit entanglement. Noting the importance of the study on multi-party quantum entanglement, our result provides a useful methodology to understand the monogamy and polygamy nature of multi-party entanglement.

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- [1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
 - [2] P. T. Landsberg and V. Vedral, Phys. Lett. A **247**, 211 (1998).
 - [3] A. K. Rajagopal and R. W. Rendell, Phys. Rev. A **72**, 022322 (2005).
 - [4] S. Abe and A. K. Rajagopal, Physica A **289**, 157 (2001).
 - [5] C. Tsallis, S. Lloyd and M. Baranger, Phys. Rev. A **63**, 042104 (2001).
 - [6] R. Rossignoli and N. Canosa, Phys. Rev. A **66**, 042306 (2002).
 - [7] J. Batle, A. R. Plastino, M. Casas and A. Plastino, J. Phys. A **35**, 10311 (2002).
 - [8] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
 - [9] B. M. Terhal, IBM J. Research and Development **48**, 71 (2004).
 - [10] J. S. Kim, G. Gour and B. C. Sanders, Contemp. Phys. **53**, 5 p. 417-432 (2012).
 - [11] L. Masanes, Phys. Rev. Lett. **102**, 140501 (2009).
 - [12] J. M. Renes and M. Grassl, Phys. Rev. A **74** 022317 (2006).
 - [13] A. J. Coleman and V. I. Yukalov, Lecture Notes in Chemistry Vol. **72** (Springer-Verlag, Berlin, 2000).
 - [14] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
 - [15] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
 - [16] T. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).
 - [17] J. S. Kim and B. C. Sanders, J. Phys. A: Math. and Theor. **43**, 445305 (2010).
 - [18] J. S. Kim, Phys. Rev. A **81**, 062328 (2010).
 - [19] J. S. Kim and B. C. Sanders, J. Phys. A: Math. and Theor. **44**, 295303 (2011).
 - [20] X. N. Zhu and S. M. Fei, Phys. Rev. A **92**, 0623425 (2015).
 - [21] For this reason, we sometimes denote

$$S_1(\rho) = S(\rho)$$
 for any quantum state ρ .
 - [22] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
 - [23] O. Cohen, Phys. Rev. Lett. **80**, 2493 (1998).
 - [24] Although $f_q(x)$ in Eq. (18) is not explicitly defined for $q = 1$, it is straightforward to check that

$$\lim_{q \rightarrow 1} f_q(x) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x^2}\right),$$
 with the binary entropy function $H(t) = -[t \log t + (1-t) \log(1-t)]$.
 - [25] G. Gour, S. Bandyopadhyay and B. C. Sanders, J. Math. Phys. **48**, 012108 (2007).
 - [26] C. S. Yu, and H. S. Song, Phys. Rev. A **76**, 022324 (2007).